CHAPTER SIX

SELECTED TOPICS FROM LINEAR ALGEBRA AND COMPUTER APPLICATIONS

Though there are many ways one could introduce linear algebra, our approach will be practical and start with matrices. A ***matrix*** is an array of numbers (a_{ij}), where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. For example, consider the simultaneous system of equations:

$$x + y = 2$$

$$2x - y = 1$$

We could transform this algebraic problem into matrix form as follows:

Let A represent the matrix for the coefficients of x and y. Therefore, $A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$. Let $B = \begin{pmatrix} x \\ y \end{pmatrix}$, the variable matrix. Let $C = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

We need to define ***matrix multiplication*** as follows:

$$\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \quad \begin{vmatrix} x \\ y \end{vmatrix} \quad = \quad \begin{vmatrix} x + y \\ 2x - y \end{vmatrix}$$

The formal way of writing the product of two matrices requires some notation:

Let $A = \begin{pmatrix} a_{11} \ldots a_{1n} \\ a_{21} \ldots a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \ldots a_{mn} \end{pmatrix}$ be an $m \times n$ matrix

This means a matrix with m rows and n columns.

To define the product of two matrices, AB, we need matrix B to have n rows. In general, matrix multiplication can only be done if the number of columns of the first matrix equals the number of rows of the second matrix.

$$\text{Let } B = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1p} \\ b_{21} & b_{22} & \ldots & b_{2p} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ b_{n1} & b_{n2} & \ldots & b_{np} \end{pmatrix}$$

Now we define the product $C = AB$ as the $m \times p$ matrix whose ik entry in the matrix is as follows:

$$C_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk} = a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + \ldots + a_{in}\, b_{nk}$$

To return to our elementary algebra example, we can write $AB = C$ as the matrix product.

$$\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \quad \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 2 \\ 1 \end{vmatrix}$$

Use the formal definition of matrix multiplication to verify the left side of the matrix equation. Be careful when substituting the appropriate subscripts.

Consider a problem to practice matrix multiplication. Let the following matrices be defined:

$$A = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} \qquad B = \begin{vmatrix} 2 & 3 & 1 \\ 4 & 2 & 0 \end{vmatrix}$$

AB is defined as a 2×3 matrix. The elements are given by:

$$C = AB = \begin{vmatrix} 2 \cdot 2 + 1 \cdot 4 & 2 \cdot 3 + 1 \cdot 2 & 2 \cdot 1 + 1 \cdot 0 \\ 3 \cdot 2 + 4 \cdot 4 & 3 \cdot 3 + 4 \cdot 2 & 3 \cdot 1 + 4 \cdot 0 \end{vmatrix}$$

$C_{11} = 4 + 4 = 8$

$C_{12} = 6 + 2 = 8$

$C_{13} = 2 + 0 = 2$

$C_{21} = 6 + 16 = 22$

$C_{22} = 9 + 8 = 17$

$C_{23} = 3 + 0 = 3$

The product matrix C is now defined as:

$$C = \begin{vmatrix} 8 & 8 & 2 \\ 22 & 17 & 3 \end{vmatrix}$$

Note that matrix BA is not defined, because B has 3 columns and A has 2 rows. Therefore, matrix multiplication is not commutative. $AB \neq BA$.

To solve matrix equations, we need another concept called the ***inverse matrix***, which in turn requires the concept of ***identity matrix***.

An ***identity matrix***, for multiplication of 2 x 2 matrices (our concern) is a matrix with the form $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$. If you multiply any 2 x 2 matrix by $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, you obtain the same matrix. For example,

$$\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix}$$

An ***inverse matrix*** for a given matrix A, written A^{-1}, is a unique matrix with the property that $A A^{-1} = A^{-1} A =$ Identity Matrix $= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ (for our 2 x 2 matrix example).

The general formula for a 2 x 2 matrix inverse is written as follows:

$$\text{If } A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad A^{-1} = \frac{1}{(a_{11}a_{22} - a_{11}a_{21})} \begin{vmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{vmatrix}$$

To illustrate the linear algebra connected with our original algebra problem, start with the matrix equation:

$$AB = C, \text{ which is written } \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 2 \\ 1 \end{vmatrix}$$

Next calculate the inverse for matrix A. First, calculate the determinant of $\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} =$

$1 \bullet (-1) - 1\,(2) = -3.$

$$A^{-1} = \frac{1}{-3} \begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix}$$

[To multiply a constant by a matrix, multiply each matrix element by the constant.]

Finally, multiply both sides of the matrix equation by A^{-1}.

$$A^{-1} \bullet A \bullet \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} x \\ y \end{vmatrix} = A^{-1} \begin{vmatrix} 2 \\ 1 \end{vmatrix}$$

$$\therefore \quad \begin{vmatrix} x \\ y \end{vmatrix} = \frac{1}{-3} \begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} \begin{vmatrix} 2 \\ 1 \end{vmatrix} = -\frac{1}{3} \begin{vmatrix} -3 \\ -3 \end{vmatrix}$$

$$\therefore \quad \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

This technique extends to 3 x 3 or n x n matrix operations, with the computer relieving us of the computational burdens.

Many statistical processes can be effectively expressed with matrix notation. To illustrate, consider the now familiar question of linear regression. Our problem of fitting the best straight line to the data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ can be interpreted as follows:

Let $y = mx + b$ be the least squares line to fit the data points (1, 2), (3, 5), and (0, 7). By least squares line we mean the line with minimum sum of squares error.

$$\sum_{i=1}^{n} \left[y_i - (mx_i + b) \right]^2$$

The formula for the solution $S = \begin{pmatrix} b \\ m \end{pmatrix}$ is:

$$S = (A^t A)^{-1} A^t y$$

To compute S, we need the concept of a ***transpose*** matrix.

The ***transpose*** of a matrix, for example of the 2 x 2 matrix $A = \begin{vmatrix} 2 & 1 \\ 5 & 6 \end{vmatrix}$, is the matrix A^t where the elements off the diagonal are interchanged. Therefore, $A^t = \begin{vmatrix} 2 & 5 \\ 1 & 6 \end{vmatrix}$. The diagonal elements of A^t remain the same as those for matrix A. We use the following matrix notation to represent the ***transpose*** for any m x n matrix B. If $B = [b_{ij}]$ is any m x n matrix, then $B^t = [b_{ji}]$ and is an n x m matrix. For example, if $B = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$, then $B^t = \begin{vmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{vmatrix}$

To compute S proceed as follows:

(1) $$A = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{pmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 0 \end{pmatrix}$$

(2) $$A^t = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \end{pmatrix}$$

(3) $$A^t A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4 \\ 4 & 10 \end{pmatrix}$$

(4) $$\det(A^t A) = 30 - 16 = 14$$

(5) $$(A^t A)^{-1} = \frac{1}{14} \begin{vmatrix} 10 & -4 \\ -4 & 3 \end{vmatrix}$$

(6) $$S = \begin{pmatrix} b \\ m \end{pmatrix} = (A^t A)^{-1} A^t Y$$

$$Y = \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$$

$$S = \frac{1}{14} \begin{vmatrix} 10 & -4 \\ -4 & 3 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \end{vmatrix} \cdot \begin{vmatrix} 2 \\ 5 \\ 7 \end{vmatrix}$$

Matrix multiplication is associative, so A (BC) = (AB) C for any matrices A, B, and C that can be multiplied. Also the constant 1/14 can be multiplied at the beginning or at the end of the process. This could be stated as, for constant d and matrices A and B, d (AB) = (dA) B, provided of course that matrices A and B can be multiplied [that is, the number of columns of A = the number of rows of B].

To complete our problem,

(7) $$\frac{1}{14} \begin{vmatrix} 10 & -4 \\ -4 & 3 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \end{vmatrix} = \frac{1}{14} \begin{vmatrix} 6 & -2 & 10 \\ -1 & 5 & -4 \end{vmatrix}$$

(8) $$S = \frac{1}{14} \begin{vmatrix} 6 & -2 & 10 \\ -1 & 5 & -4 \end{vmatrix} \begin{vmatrix} 2 \\ 5 \\ 7 \end{vmatrix} = \frac{1}{14} \begin{vmatrix} 72 \\ -5 \end{vmatrix}$$

(9) $$S = \begin{pmatrix} b \\ m \end{pmatrix} = \begin{vmatrix} 72/14 \\ -5/14 \end{vmatrix} = \begin{vmatrix} 5.14 \\ -.36 \end{vmatrix}$$

(10) Our desired least squares line is: $y = -.36x + 5.14$

We could extend this process to quadratic regression as follows:

Let $y = a + b\,x + c\,x^2$ be the desired quadratic equation to data points (x_1, y_1), (x_2, y_2), . . . , (x_n, y_n). We let:

$$y = \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{pmatrix} \qquad A = \begin{pmatrix} 1 & x_1 & x_{\cdot}^2 \\ 1 & x_2 & x_2^2 \\ . & & \\ . & & \\ 1 & x_n & x_n^2 \end{pmatrix}$$

$$\text{and } S = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Then $S = (A^t A)^{-1} A^t y$.

The topics that have just been introduced enable the reader to understand some but not nearly all of the preliminary concepts underlying the International Mathematical Modeling Contest problem that is presented together with an exemplary solution at the end of this text. The problem, "Midge Classification" for MCM 1989, could initially be considered as a regression problem. The reader is advised to inspect the problem and to peruse the exemplary solution.

The outstanding solution went well beyond the simplistic analysis that the problem was simply an exercise in linear regression. In order to understand the outstanding paper, the student is directed to an advanced statistics text, such as *SPSS Advanced Statistics Student Guide*, which clearly presents the appropriate statistical tool to tackle this question - ***discriminant analysis***. The explanation of discriminant analysis will use both the concepts of matrices and matrix inverses. Also, the brilliant paper used a numerical analysis procedure to perform a goodness of fit test for a joint bivariate normal density. This advanced goodness of fit test is typically an optional and consequently rarely encountered topic in undergraduate (or for that matter, graduate) level Numerical Analysis.

Remember that this paper is one of the finest completed in the world. Also be assured that whether your paper is judged as "outstanding" or simply "successful participant," the contest is designed for you to experience the fruits of teamwork, to work on an applied problem in the way that you likely will in the future - as a member of a team. And along the way you will have learned a great deal of one of the most useful and promising tools for research in the 2000s and beyond - mathematical modeling.

Advanced Topics

The preceding work has gotten you started in a new and exciting discipline called mathematical modeling. We next present two computer applications of modeling, reprinted with the permission of *Byte*. The book continues with two outstanding papers in the final chapter, one using statistics and the other involving graph theory. We recommend that you obtain advanced texts that allow you to make the leap between this introductory text and the research type problems that are typical in the Mathematical Modeling Competition. The references below should assist you with bridging the considerable gap between this introductory manual and the Mathematical Modeling Contest problems.

References

Statistics

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Graph Theory

1. Chartrand, G. *Introductory Graph Theory* (New York: Dover, 1977).

2. Harary, F. *Graph Theory* (Reading: Addison Wesley, 1969).

3. Malkevitch, J. and Meyer, W. *Graphs, Models and Finite Mathematics* (Englewood Cliffs: Prentice Hall, 1974).

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